Pipe-line Design for Non-Newtonian Fluids in Streamline Flow

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A method is presented for determining the flow-rate—pressure-gradient relationship for the streamline flow of the large class of non-Newtonian, nonthixotropic fluids to which the Powell-Eyring equation applies. The general procedure and assumptions required in developing this method are the same as used in deriving the Hagen-Poiseuille equation except that the Powell-Eyring equation is used in place of Newton's equation to relate shear stress to shear rate.

The mehod can be used to predict pipe-flow pressure gradients from both precision viscometer data and pipe-flow data. Its applicability is demonstrated for three typical non-Newtonian fluids, 3% carboxymethylcellulose in water, 15% napalm in kerosene, and 33% hydrated lime in water.

When used with pipe-flow data, it resembles the method of Alves and associates, compensating for the inconvenience of an additional step in calculation procedure by providing a means of extrapolating beyond the range of the experimental data.

The relationships developed facilitate the application of shear-stress—shear-rate data in the design of conduits for non-Newtonian fluids.

The basic equation used for expressing the volumetric flow rate of a fluid in a cylindrical conduit is

$$q = \int_0^R 2\pi r u \, dr \qquad (1)$$

where the assumption is made that u is a function of r only. For Newtonian fluids in streamline flow, velocity is related to radial position by the elimination of shear stress between Newton's postulate that shear stress is directly proportional to shear rate,

$$\sigma = \mu \frac{du}{dx} \qquad (2)$$

and a force balance on a cylindrical element of fluid (radius r, length dL) coaxial with the conduit,

$$\sigma = -\frac{r}{2} \frac{dp}{dL} \qquad (3)$$

The relationship thus obtained is substituted into Equation (1), and the integration is performed to obtain the familiar Hagen-Poiseuille isothermal flow equation,

$$q = \frac{\pi R^4}{8\mu}\left(-\frac{dp}{dL}\right) \qquad (4)$$

Non-Newtonian fluids do not conform to Equations (2) and (4), as the ratio of shear stress to shear rate varies with the shear rate and in some cases is also a function of the flow history of the fluid (thixotropic fluids).

In the case of non-Newtonian flow in pipe most investigators have sought substitutes for Equation (4) by one of two general methods. In the first an empirical equation representing the experimental shear-stress–shear-rate data is used in place of Equation (2) in the derivation as outlined (reference *4*, for example). Though this method is applicable to some fluids, it is seldom that empirical equations of conveniently simple form fit the data over large ranges of shear rate with the desired accuracy. The second general method involves correlating suitable flow parameters sufficiently general in character that they may be used with laboratory tube-flow data or rotational viscometer data to predict flow in commercial piep (references *1*, *2*, and *5*, for example).

The method developed in this study embodies features of both general methods. An equation (the Powell-Eyring equation) which accurately represents the shear-stress–shear-rate data for a large number of non-Newtonian fluids over a wide shear-rate range is used in deriving a relationship between the pipe-flow variables (flow rate, pressure gradient, and pipe radius) and rheological properties of the fluid in terms of dimensionless groups. The development and application of this method follows.

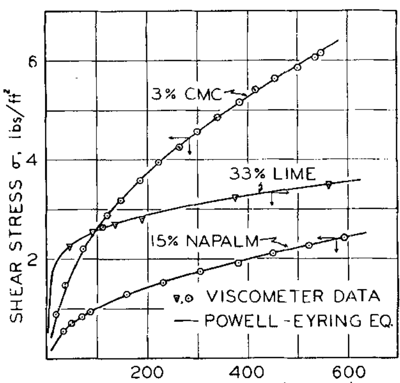

Fig. 1. Comparison of predicted shear data with experimental data.

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SHEAR-STRESS—SHEAR-RATE RELATIONSHIPS

From the theory of absolute reaction rates Eyring and his associates (*3*) developed the hyperbolic sine law relating shear stress to shear rate:

$$\sigma = \frac{1}{B} \sinh^{-1}\left(\frac{1}{A}\frac{du}{dx}\right) \qquad (5)$$

Powell and Eyring (*7*) subsequently suggested a modification, called here the Powell-Eyring equation, which they and others (*9*) have applied to non-Newtonian fluids:

$$\sigma = \mu\left(\frac{du}{dx}\right) + \frac{1}{B}\sinh^{-1}\left(\frac{1}{A}\frac{du}{dx}\right) \qquad (6)$$

The constants A, B, and μ are characteristic of the fluid.

Experimental shear-stress–shear-rate data for 3% carboxymethylcellulose in water, 15% napalm in kerosene, and 33% hydrated lime in water and the corresponding curves calculated by means of the Powell-Eyring equation are compared in Figure 1. In the case of 15% napalm in kerosene, data were taken to shear rates of 2,400 sec.$^{-1}$, and the curve of Figure 1 was calculated from the Powell-Eyring equation, which fits the data to this relatively high shear rate. The agreement is within experimental error and is typical of that obtained by use of the rotational viscometer described below (*9, 10*). The constants A, B, and μ used to plot the curves in Figure 1 were determined from the smoothed experimental data, as demonstrated in Illustration 1.

The flow behavior of the lime slurry, often considered a Bingham-plastic fluid, is as well described by the equation as is the flow behavior of the typical pseudoplastic fluids. It is likely that many fluids, ordinarily regarded as Bingham plastics, are in reality extreme pseudoplastics and have flow characteristics that can be accurately represented by the Powell-Eyring equation. Although Equation (5) fits the data of this study fairly well for a limited range of variables, agreement is poor for extended ranges.

PIPE-FLOW RELATIONS

The general procedure followed in deriving the Hagen-Poiseuille Equation (4) is used in this study to derive an equivalent pipe-flow-design equation for non-Newtonian fluids. In place of Newton's Equation (2) the Powell-Eyring Equation (6) is used. The result may be represented as

$$\alpha = f(\beta, \gamma) \tag{7}$$

where α, β, and γ are dimensionless groups defined as follows:

$$\alpha = \frac{q}{AR^3} \tag{8}$$

$$\beta = -\frac{BR}{2}\frac{dp}{dL} \tag{9}$$

$$\gamma = A\,B\,\mu \tag{10}$$

These groups would be predicted by application of dimensional analysis to the pipe-flow problem, with flow rate q, pressure gradient $(-dp/dL)$, pipe radius R, and the fluid properties A, B, and μ as variables.

The complex equation obtained by the analytical solution and represented by Equation (7) was used to prepare Figure 2, the basic design chart.*

Equation (7) may be regarded as a general equation from which both Equation (4) and its analogue, derived with Equation (5) in place of Equation (2), are special forms. In either of two cases, $\gamma >> 1$ or β very large with $\gamma \neq 0$, the explicit form of Equation (7) reduces to

$$\alpha = \frac{\pi\beta}{4\gamma},$$

which is the Hagen-Poiseuille Equation (4) in terms of dimensionless groups. When $\gamma = 0$ (i.e., $\mu = 0$), the explicit form of Equation (7) becomes

$$\alpha = \frac{2\pi}{\beta^3}\left[\left(1+\frac{\beta^2}{2}\right)\cosh\beta - \beta\sinh\beta - 1\right] \tag{11}$$

*Tabulated values of α, β, and γ have been calculated from the explicit form of Equation (7). These data, the explicit form of Equation (7), (Table 4), and shear data (Table 5) may be obtained from the American Documentation Institute, Auxiliary Publications Photo Duplications Service, Library of Congress, Washington 25, D. C., as document 4745 for $1.25 for photoprints or microfilm.

the analogue of Equation (4) based on the hyperbolic sine law (5).

Equation (6) and Figure 2 serve as the basic relationships for the design of pipe-flow systems.

DESIGN OF PIPE-FLOW SYSTEMS

The quantities that must be determined in the design of pipe-flow systems are the pipe radius R, pressure gradient $(-dp/dL)$, and flow rate q, the value of one being calculated from known or assumed values of the other two. The calculation using Figure 2 requires knowledge of the characteristic constants of the fluid, A, B, μ, just as for Newtonian flow use of Equation (4) requires that viscosity be known. In both cases the fluid characteristics may be determined in either of two ways. First, shear-stress–shear-rate data may be used with Equation (2) for Newtonian fluids or with Equation (6) for non-Newtonian fluids to which it applies. Second, tube-flow data, as obtained in capillary or larger tubes, may be used with Equation (4) for Newtonian fluids or with Figure 2 for non-Newtonian fluids. The methods are explained in the following illustrations.

Illustration 1. Shear stress — shear-rate Data Given

PROBLEM: To calculate the pressure gradient in a 1½-in. pipe ($R = 0.0647$ ft.) when 15% napalm in kerosene flows through it at a flow rate of 0.0254 cu.ft./sec., given the rotational viscometer data of Figure 1 at the same temperature:

a. These sets of shear data are selected from Figure 1:

du/dr, sec.$^{-1}$	σ, lb. force/sq.ft.
452	2.11
104	1.05
30	0.47

b. Each of the three sets of viscometer data is substituted into Equation (6) and the three equations obtained are solved simultaneously for constants:

$A = 32.1$ sec.$^{-1}$
$B = 1.98$ sq.ft./lb. force
$\mu = 0.00093$ (lb. force) (sec.) /sq.ft.

c. From Equation (10), $\gamma = 0.059$
d. From Equation (8), $\alpha = 2.92$
e. From these values and Figure 2, $\beta = 2.5$
f. From Equation (9), $-dp/dL = 39.0$ lb. force/cu.ft.

The value obtained experimentally is 39.5 lb. force/cu.ft.

Illustration 2. Pipe-flow Data Given

PROBLEM: Same as in Illustration 1, given, in place of the viscometer data, the experimental ⅞-in. tube-flow data in Table 2.

a. For each q and the corresponding dp/dL, $A\alpha = q/R^3$ and $\beta/B = -(R/2)(dp/dL)$ are calculated by use of the ⅞-in. tube radius; $R = 0.0326$ ft. A data plot of β/B as ordinate and $A\alpha$ as abscissa is prepared on logarithmic paper having the same scale as Figure 2.

b. The data plot is placed on Figure 2 and shifted, with the two sets of coordinate axes parallel, until the best match between the lines on Figure 2 and the data plot is obtained. The value of γ is read from the line on Figure 2 matched by the data plot. In this case $\gamma = 0.075$.

c. Any convenient point is taken on Figure 2, and the values of α and β are read. From the point on the superimposed data plot, immediately above the point taken on Figure 2, the values of $A\alpha$ and β/B are read.

d. The constants A, B, and μ are calculated as follows:

$A = (A\alpha)_{data}/\alpha_{Fig.\,2} = 29.5$ sec.$^{-1}$
$B = \beta_{Fig.\,2}/(\beta/B)_{data} = 2.09$ sq. ft./lb. force
$\mu = \gamma/AB = 0.00122$ (lb. force) (sec.) /sq.ft.

e. The remainder of the procedure is as in Illustration 1, starting at step *c*; the constants shown above and $R = 0.0647$ ft. are used:

$\alpha = 3.18$
$\beta = 2.65$
$-dp/dL = 39.3$ lb./cu.ft., as compared to 39.5, found experimentally.

The calculation of q when $-dp/dL$ is known proceeds in a similar manner. The situation is more complicated, however, when R is desired, because R appears in both α and β. In this case the constants A, B, and γ are found as in the foregoing illustrations. For each of two assumed values of R and the given values of q and dp/dL, α and β are calculated from Equations (8) and (9). The two resulting (α, β) points are located on Figure 2. The straight line joining these points is extended until it intersects the line characterized by the known value of γ. The value of α (or β) corresponding to this intersection may be substituted into Equation (8) or (9) to yield the desired value of R.

It will be noted that the values of A, B, μ in Illustration 1 differ from those in Illustration 2, though the same fluid was considered. The two calculated values of $-dp/dL$ are, however, in good agreement. If in the case of the curve-matching method of Illustration 2, the data plot matches two adjacent γ lines of Figure 2 equally well, either γ line may be used, although the calculated constants will differ. Owing to compensating differences within the sets of constants, the resulting Powell-Eyring equations will fit the shear data about equally well. Consequently the calculated

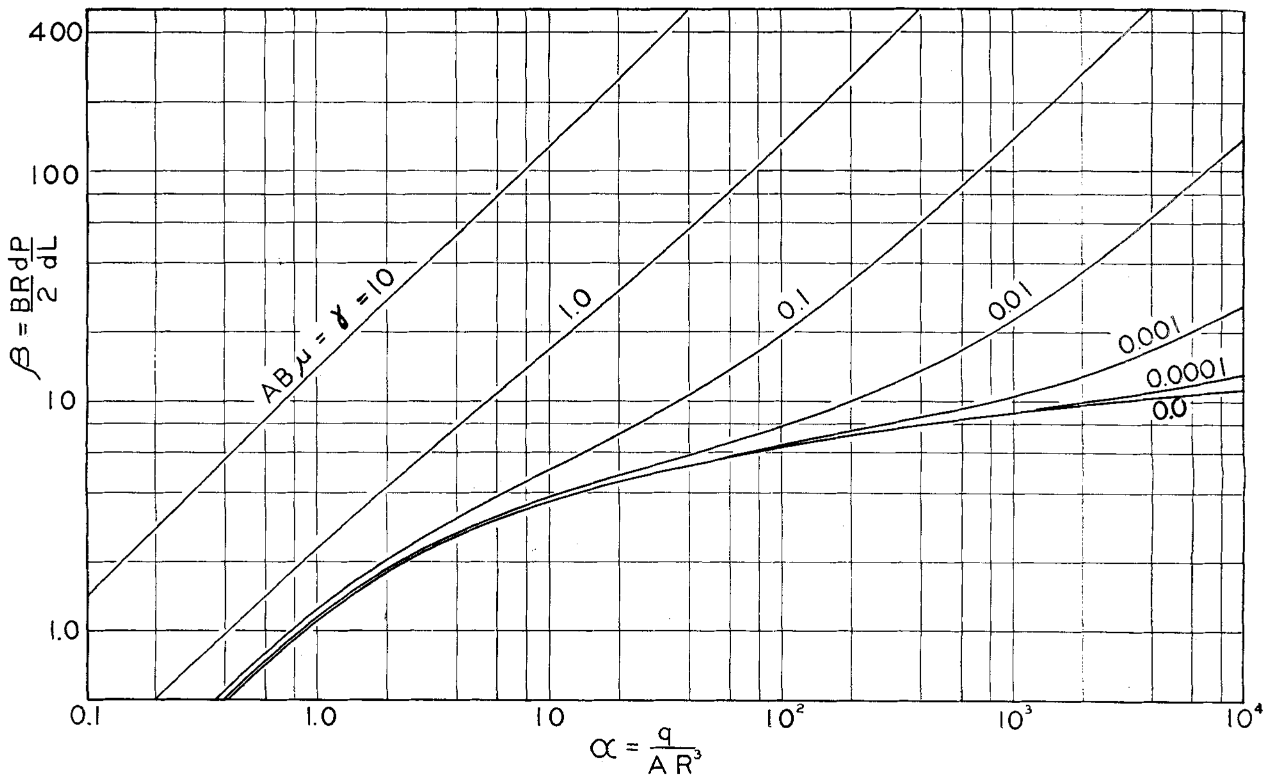

Fig. 2. Non-Newtonian pipe-flow prediction graph.

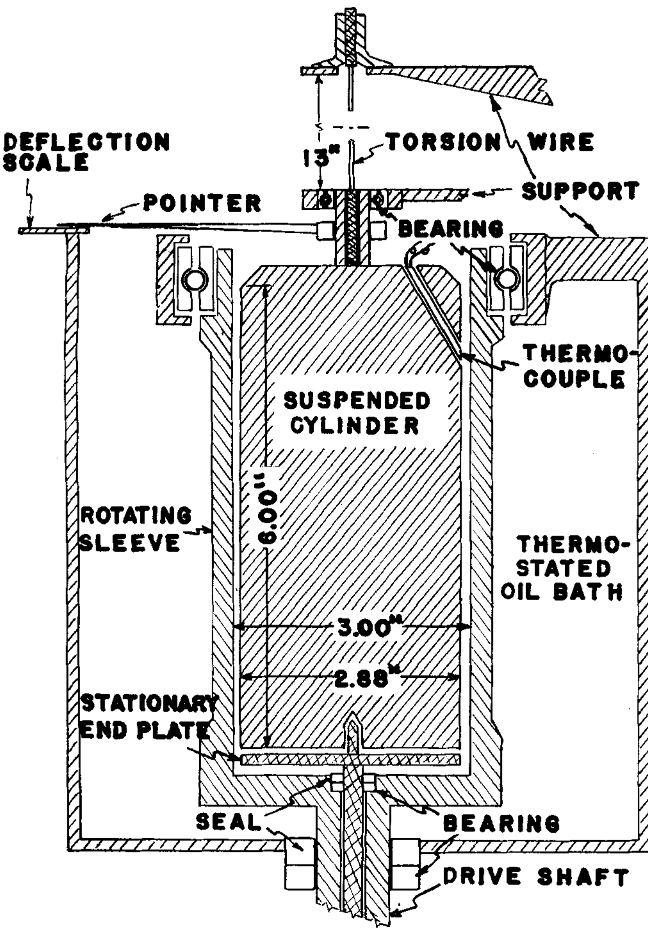

Fig. 3. Rotational viscometer.

values of q, $-dp/dL$, or R will be very nearly the same. Greater accuracy may be expected by averaging results for two adjacent, equally well-matched γ lines, particularly if the desired α and β values require extrapolation far beyond experimental pipe-flow data.

EXPERIMENTAL EQUIPMENT

Experimental measurements were taken in the rotational-type viscometer and the pipe-flow system shown in Figures 3 and 4. The equipment is described in detail in reference *10*.

Rotational Viscometer

Essential features of several rotational viscometers are discussed in the literature (*1*). The instrument used in this investigation consists of a metal cylinder suspended concentrically in a metal cylindrical sleeve by means of an interchangeable torsion wire. The clearance between the wall of the cylinder and the wall of the sleeve (0.061 in.) forms an annular space into which the fluid to be tested is placed. The instrument contains a stationary disk at the bottom of the suspended cylinder to eliminate the transmission of torque to the bottom of the suspended cylinder. The sleeve is mounted in such a manner that it can be made to rotate at controlled speeds.

The viscometer is mounted in a thermostated oil bath. The temperature of the test fluid is measured by a thermocouple mounted in the sus-

TABLE 1.—PRESSURE GRADIENT PREDICTIONS, 3% CARBOXYMETHYCELLULOSE IN WATER

⅞-in. iron tube (R=0.0322 ft.)				1½-in. pipe (R=0.0647 ft.)		
q	$-dp/dL$, lb force/cu. ft.			q	$-dp/dL$, lb. force/cu. ft.	
cu. ft./sec.	Obs.	Calc.*	Calc.†	cu. ft./sec.	Obs.	Calc.*
0.00058	74.1	59.2	64.5	0.0089	51.1	48.9
0.00129	113.2	112.4	113	0.0159	71.1	72.8
0.00201	151	150.3	154	0.0275	97.6	100.8
				0.0304	103.8	107.5
				0.0481	132.5	137.2
				0.0501	134.6	139.6
Percentage mean deviation		(7.2)	(5.0)			(3.5)

*Calculated from viscometer data. A=26 sec.$^{-1}$, B=0.934 sq. ft./lb. force, μ=0.00399 (lb. force) (sec.)/sq. ft.
†Calculated from 1½-in. pipe-flow data. A=16.9 sec.$^{-1}$, B=1.18 sq. ft./lb. force, μ=0.00501 (lb. force) (sec.)/sq. ft.

TABLE 2.—PRESSURE GRADIENT PREDICTIONS, 15% NAPALM IN KEROSENE

⅞-in. copper tube (R=0.0326 ft.)				1½-in. pipe (R=0.0647 ft.)				
q,	$-dp/dL$, lb. force/cu. ft.			q,	$-dp/dL$, lb. force/cu. ft.			
cu. ft./sec.	Obs.	Calc.*	Calc.‡	cu. ft./sec.	Obs.	Calc.*	Calc.†	Calc.‡
0.00120	40.9	39.7	46.0	0.00778	18.9	17.3	17.6	21.2
0.00197	56.8	56.3	58.7	0.01724	31.5	30.6	31.0	31.3
0.00258	65.6	66.5	67.0	0.0254	39.5	39.0	39.3	37.8
0.00584	107.6	103.7	99.7	0.0431	52.3	51.0	52.6	48.8
0.00806	127.4	122.5	116.8	0.0516	56.6	55.7	57.4	53.4
0.01007	144.6	136.2	130.0	0.0658	62.1	63.2	65.2	60.1
Percentage mean deviation		(3.1)	(7.3)			(3.1)	(2.7)	(5.4)

*Calculated from viscometer data Equation (6), and Figure 2. A=32.1 sec.$^{-1}$, B=1.98 sq. ft./lb. force, μ=0.00093 (lb. force) (sec.)/sq. ft.
†Calculated from ⅞-in. tube data and Figure 2. A=29.5 sec.$^{-1}$, B=2.09 sq. ft./lb. force, μ=0.00122 (lb. force) (sec.)/sq. ft.
‡Calculated from viscometer data and Equations (12) and (13).

TABLE 3.—PRESSURE GRADIENT PPEDICTIONS, 33% LIME SLURRY

2-in. Pipe (R=0.0865 ft.)			1½-in. Pipe (R=0.0647 ft.)			⅞-in. Copper tube (R=0.0326 ft.)		
q	$-dp/dL$, lb. force/cu. ft.		q	$-dp/dL$, lb. force/cu. ft.		q	$-dp/dL$, lb. force/cu. ft.	
Cu. ft./sec.	Obs.	Calc.*	Cu. ft./sec.	Obs.	Calc.*	Cu. ft./sec.	Obs.	Calc.*
0.0269	6.24	6.24	0.0255	11.33	9.42	0.0136	24.5	24.6
0.0341	6.88	6.43	0.0261	10.88	9.48	0.0193	27.2	26.7
0.0457	7.45	6.75	0.0327	11.52	9.93	0.0240	29.0	28.5
0.0634	7.77	7.13	0.0385	11.71	10.31			
0.0892	7.90	7.65	0.0492	12.22	10.69			
0.1118	8.15	7.90	0.0616	13.11	11.14			
Percentage mean deviation		(5.1)			(13.8)			(1.1)

*Calculated from viscometer data. A=0.191 sec. $^{-1}$, B=27.6 sq. ft./lb. force, μ=6.16 (10^{-5}) (lb. force) (sec.)/sq. ft.

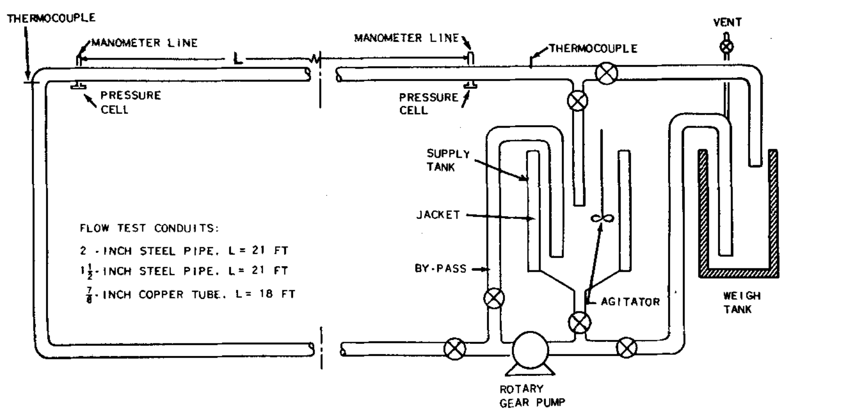

Fig. 4. Experimental pipe-flow system.

pended cylinder wall, as shown in Figure 3. The junction, flush with the cylinder surface, is insulated from the metal.

In a given test the sleeve is rotated at controlled speeds. The moment exerted by the sheared fluid on the inner cylinder causes it to rotate until the moment is balanced by the torque of the wire. Shearing stress is calculated from the angular displacement of the cylinder, the dimensions of the cylinder, and calibration data for the torsion wire. The rate of shear is calculated from the rotational speed of the sleeve and dimensions of the apparatus.

Pipe-flow System

The pipe-flow system, sketched in Figure 4, provides a 21-ft. test section preceded and followed by 8-ft. calming sections for each of the following conduits: ⅞-in. tube, 1½-in. black-iron pipe, and 2-in. galvanized-iron pipe. Temperature measurements, taken by means of thermocouples located at the ends of the pipes, are recorded by a recording potentiometer. Pressures at each end of the 21-ft. test section are determined by means of both mercuryfilled manometers and Baldwin (type EMB) fluid pressure cells. The output electromotive forces of the cells are also recorded by the potentiometer. Flow measurements are made by weighing the quantity of fluid flowing through the conduit in use during a measured time interval. The flow rates are varied by short-circuiting pumped fluid back to the feed-storage vessel. Conduit diameters were calculated from measured conduit volumes, and lengths were checked by use of Equation (4) and conduit-flow data for sucrose solutions.

DISCUSSION OF RESULTS

Experimental flow rates and pressure gradients for three non-Newtonian fluids together with pressure gradients calculated by the methods of Illustrations 1 and 2 are presented in Tables 1, 2, and 3. The shear-stress–shear-rate relationships for these fluids are shown in Figure 1. The mean deviation of predicted pressure gradients from experimental values is seen to be less than 4% for 15% napalm in kerosene. Similar accuracy can be attributed to predictions for 3% carboxymethylcellulose in water, except for very low flow-rates in the ⅞-in. tube.

Pressure-gradient predictions for the 33% lime slurry are less successful. Reproducible rotational-viscometer data were not obtained for the lime slurry, presumably because of settling of the suspended lime particles. The shear-stress–shear-rate data shown on Figure 1 were taken to be as representative of the unsettled slurry as any data taken and were used to obtain the constants A, B, and μ employed in the calculations. Under the circumstances the agreement between calculated and observed pressure gradients in the 2-in. pipe and ⅞-in. tube is much better than might be expected.

The design method presented is limited to use with fluids whose behavior does not depend on their flow history and to fluids that conform to the Powell-Eyring equation. The latter limitation is not severe, because of the versatility of the Powell-Eyring equation. As explained above, it represents very well shear data for 33% lime slurry, which is often considered to be a Bingham plastic, as well as shear data for typical pseudoplastics. Also, as demonstrated in the case of 15% napalm in kerosene, the Powell-Eyring equation fits shear-stress–shear-rate data accurately over wide ranges.

Extrapolation can be made within the range of Powell-Eyring equation applicability. For instance, the flow calculations in Table 3 for lime slurry in a ⅞-in. tube represent extrapolations beyond the experimental shearing stresses and shearing rates up to 35 and 300% respectively.

Alves and associates (*1, 2*) have recommended a design method in which tube-flow data are plotted as $q/\pi R^3$ vs. $(R/2)(-dp/dL)$ for the fluid of interest. This graph, which is essentially the same as the data plot described in Illustration 2, is then used to design pipe-flow systems. The advantage of using Figure 2 with the data plot is that it provides a basis for extrapolation to values of $q/\pi R^3$ and $(R/2)(-dp/dL)$ considerably beyond the range of the experimental data.

As previously mentioned, Eyring's hyperbolic sine law [Equation (5)] approximates the shear data of this study over small ranges of shear rate. Accordingly, for approximate results, Equations (11) and (5) with shear data or Figure 2 with $\gamma = 0$ and pipe-flow data may be used with some saving in time. When these procedures were followed with data shown in Figure 1 and Tables 1, 2, and 3, the values of dp/dL calculated were generally in error by less than 20% (*10*). For fluids conforming closely to the hyperbolic sine law in the range of shear rates of interest, these procedures should give results as accurate as the data used.

However, if approximate results will suffice and if data extrapolation is not required, the power Equations (6) (8),

$$\frac{du}{dx} = k\,\sigma^n \tag{12}$$

and the corresponding analogue of Equations (4) (8),

$$q = \frac{\pi k}{3+n} R^{3+n} \left(\frac{-dp}{2dL}\right)^n \tag{13}$$

together with shear data, are more convenient to use than the Eyring equation. Typical results obtained by use of Equations (12) and (13) with shear data are compared with experimental results in Table 2. Similar accuracy may also be obtained by means of Equation (13) with constants determined from pipe-flow data if appreciable shear-rate extrapolation is not involved.

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NOTATION

(Units are given in the engineering system, but any consistent system may be used.)

- A = constant characteristic of fluid, defined by Equation (6), sec.$^{-1}$
- B = constant characteristic of fluid, defined by Equation (6), sq.ft./lb. force
- f = function of
- k = constant in power equation
- L = distance along conduit in direction of flow, ft.
- n = constant in power equation
- p = static pressure of fluid, lb. force/sq. ft.
- q = volumetric flow rate, cu.ft./sec.
- r = radial distance from pipe axis, or from viscometer axis, ft.
- R = radius of pipe or tube, ft.
- u = local velocity of fluid, ft./sec.
- du/dx = shearing rate, sec.$^{-1}$ *Note*: $dx = dr$ for rotating cylinder rotational viscometer, $dx = -dr$ for pipe flow
- α, β, γ = dimensionless groups, defined by Equations (8), (9), and (10)
- μ = constant characteristic of fluid, defined by Equation (6), or viscosity, defined by Equation (3), (lb. force/(sec.)/sq. ft.
- σ = shearing stress, lb. force/sq. ft.

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